Solving first order ODEs in Python: Part 1

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First order ODEs and IVP

We want to solve a **first order ordinary differential equation** for a function u of variable $t \in [0, \infty)$ which satisfies the equation of the form

$$\frac{du}{dt} = f(u, t) \text{ for } t \in (0, \infty).$$

If $u(0) = u_0$ is given, we call the problem an **initial value problem**. A solution of IVP exists on a neighborhood of t and is unique given f is Lipshitz continuous. (Picard–Lindelöf theorem)

Example problems

1. $\dot{u} = u$: analytic solution $u(t) = u_0 e^t$
2. $\dot{u} = 1 - u^2$: analytic solution $u(t) = \frac{(u_0+1)e^{2t}+u_0-1}{(u_0+1)e^{2t}-(u_0-1)}$

   Different behavior when u_0 > 1, =1, <1
3. A model of fishery $\dot{u} = u(1-u) - h$
4. Improved model of a fishery $\dot{u} = u(1-u) - h\frac{u}{a+u}$
5. Plane dynamics $\dot{x} = y, \dot{y} = -4x$

Plot a 2D phase portrait in Python

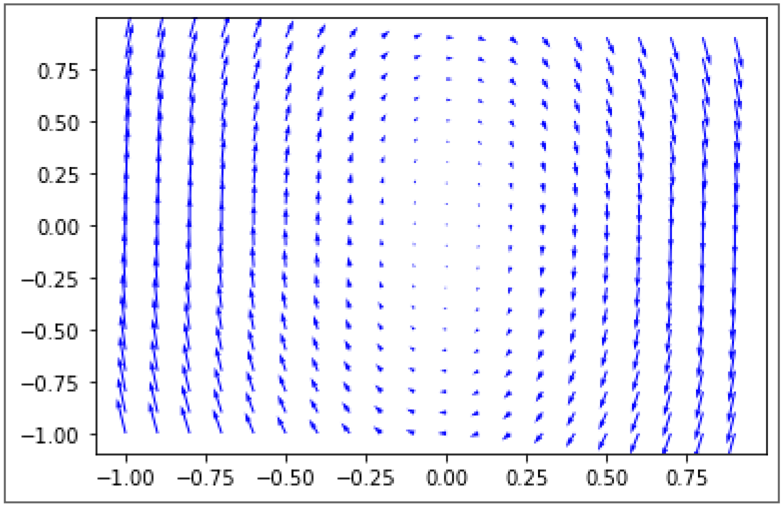
First, generate 2D gridpoints using `numpy.meshgrid` and then plot vector field using `matplotlib.pyplot.quiver`.

```
X, Y = numpy.meshgrid(x, y)
matplotlib.pyplot.quiver(X, Y, U, V)
```

See example 5.

In [11]:

```
def f5(x, y):
    return y, -4*x

import numpy as np
import matplotlib.pyplot as plt

X, Y = np.meshgrid(np.arange(-1,1, 0.1), np.arange(-1,1, 0.1))
U, V = f5(X, Y)
plt.quiver(X, Y, U, V, color='blue')
plt.show()
```



Discretization of equation and domain

To solve IVP using computer, we **discretize** both the domain $t \in [0, \infty)$ and the equation. Then, find a solution to this discretized problem which is an approximate solution to the original problem. First, discretize the domain:

- Let $t_n = n * dt$ for a constant dt for each $n = 0, 1, 2, \cdots$
- Here, dt is the time step.
- Let the approximated solution $u_n \approx u(t_n)$.

Then, discretize the equation:

- Forward Euler: Approximate $\frac{du}{dt} \approx \frac{u_{n+1} - u_n}{dt}$.
    - The discretized equation is $u_{n+1} = u_n + dt * f(u_n)$.
    - Given an initial condition u_0, this explicit equation can be solved iteratively.
    - Order of approximation: $O(dt)$
- Backward Euler, ...

Python code for Forward Euler

In [12]:

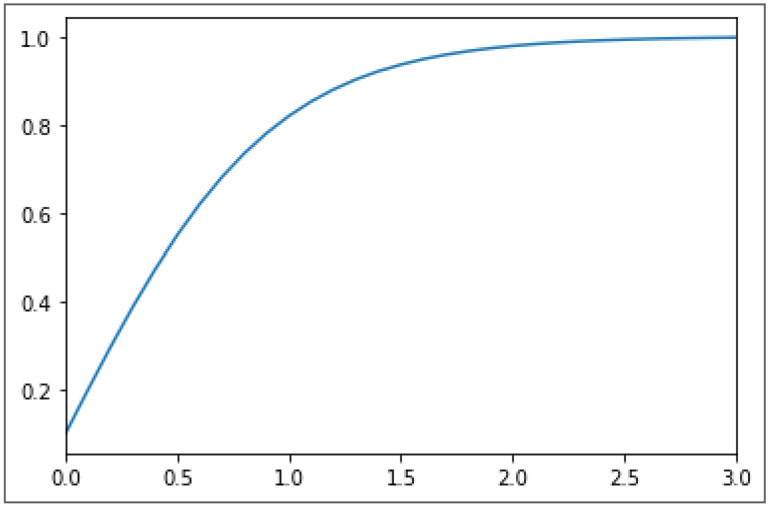
```
def FE(f, dt, tn, u0, parms=None):
    """Solve u=f(u; parms)
       f: callable, dt: float, tn: float, u0: float, parms: iterable
       u, t: numpy 1D array of length n+1"""
    import numpy as np

    n = int(tn/dt)
    u = np.zeros((n+1))
    t = np.arange(start=0, stop=tn+dt, step=dt)
    u[0]=u0

    for i in range(n):
        u[i+1] = u[i] + dt*f(u[i], parms)

    return u, t
```

In [13]:

```
# define vector fields
def f1(x, parms=None):
    return x
def f2(x, parms=None):  # Try IC > 1, =1, <1
    return 1-x**2
def f3(x, parms): # a model of a fishery
    h = parms[0]
    return x*(1-x)-h
def f4(x, parms): # Improved model of a fishery
    h = parms[0]
    a = parms[1]
    return x*(1-x)-h*x/(a+x)
```

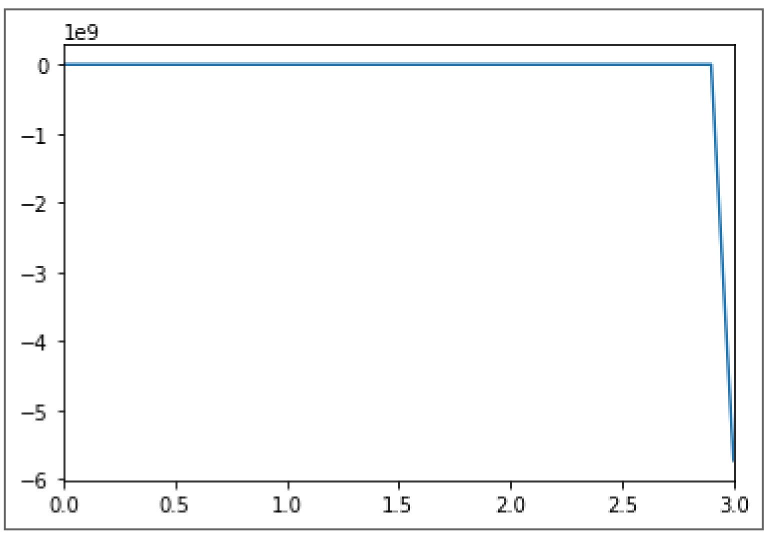
In [17]:

```
import matplotlib.pyplot as plt

u0 = 0.1
tn = 3
dt = 0.1

u,t = FE(f2, dt, tn, u0)  # modify to test on f2 with u0>1, =1, <1
plt.plot(t, u)
plt.xlim((0,tn))
plt.show()
```



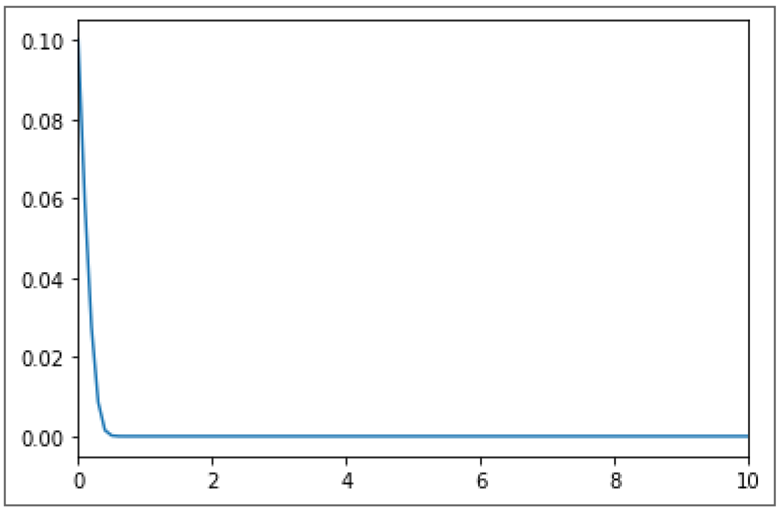
In [20]:

```
# A model of fishery: f3
"""h<1/4 : two scenarios for IC<<1, IC>smaller equilibrium point
   h>1/4 : decrease anyway"""
import matplotlib.pyplot as plt

h = 0.3
u0 = 0.1
tn = 3
dt = 0.1

u,t = FE(f3, dt, tn, u0, (h,))
plt.plot(t, u)
plt.xlim((0,tn))
plt.show()
```



In [24]:

```
# Improved model of fishery: f4
"""Check the dynamics near 0
   0 is stable if a>1 & h>a
   0 is unstable if h<a
   0 is stable if a<1 & h>a"""
import matplotlib.pyplot as plt
a = 0.1
h = 1
u0 = 0.1
tn = 10

u,t = FE(f4, 0.1, tn, u0, (h, a))
plt.plot(t, u)
plt.xlim((0,tn))
plt.show()
```



Python module for Higher order methdods

Compare the order of accuracies of different methods.

$$
\begin{aligned}
Forward Euler &\quad O(h) \\
Backward Euler &\quad O(h) \\
Central difference &\quad O(h^2) \\
RK23 &\quad O(h^2) \\
RK45 &\quad O(h^4)
\end{aligned}
$$

Test higher order methods using the Python module:

`scipy.integrate.solve_ivp`.

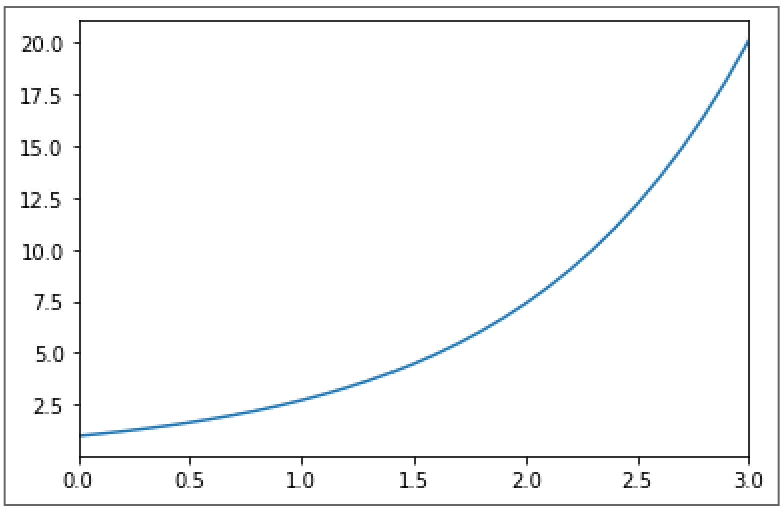
```
scipy.integrate.solve_ivp(fun, t_span, y0, method =
'RK45')
```

In [25]:

```
"""define vector fields to provide input to solve_ivp
   solve_ivp assumes solving general non-autonumous ODEs with variable t"""
def f1_(t, x, parms=None):
    return x
def f2_(t, x, parms=None):  # Try IC > 1, =1, <1
    return 1-x**2
def f3_(t, x, h): # a model of a fishery
    return x*(1-x)-h
def f4_(t, x, h, a): # Improved model of a fishery
    return x*(1-x)-h*x/(a+x)
def f5_(t, z):
    x, y = z
    return y, -4*x
```

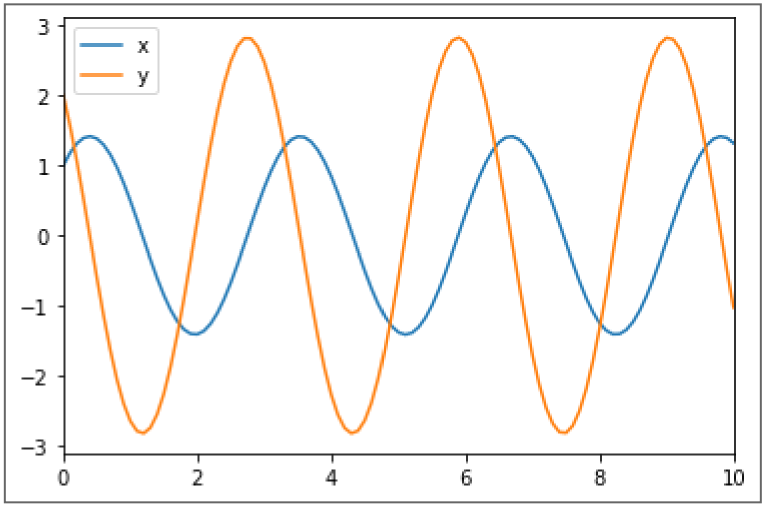
In [26]:

```
import numpy as np
import scipy.integrate
import matplotlib.pyplot as plt

dt=0.1
tn=3
u0 = 1

sol = scipy.integrate.solve_ivp(f1_, (0, tn), (u0,), method = 'RK45', t_eval = np.arange(0, tn+dt, dt))

plt.plot(sol.t, sol.y[0])
plt.xlim((0,tn))
plt.show()
```



In [4]:

```
# 2D example: periodic
u0 = 1
v0 = 2
tn = 10
dt = 0.1

sol = scipy.integrate.solve_ivp(f5_, (0, tn), (u0,v0), method = 'RK45', t_eval = np.arange(0, tn+dt, dt))

plt.plot(sol.t, sol.y[0])
plt.plot(sol.t, sol.y[1])
plt.xlim((0,tn))
plt.legend(('x', 'y'))
plt.show()
```



Remaining things that you could do

- Take an example including parameters and see the bifurcation behaviors.
- Improve the models of fisheries.
- Take your owvn domain-specific problem, solve it and interpret the result.

Summary

- discussed numerical methods to solve IVPs for ODEs,
- learned to implement / use an ODE solver in Python,
- and tested to some examples

In the next video, we would handle more complicated problems such as SIR, Lorenz equations using `solve_ivp`.