Lorenz Equations for Atmospheric Convection Modeling

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Outline of the study

- Lorenz [1] introduced a dynamical system, the *Lorenz equations* in 1963 which describes the Earth's atmospheric convection.
- Using governing equations in 2D hydrodynamics, the steps of Lorenz are followed to derive the Lorenz equations from an abstract climate model.
- Then, Lorenz equations are analyzed by its equilibrium solutions and illustrated by individual examples.
- In the final discussion, application to the atmospheric convection modeling is stated and a different approach to handle the problem is proposed.

The setting from Lorenz

Rayleigh-Bernard flow

- The Earth's atmosphere is assumed an incompressible fluid situated between two horizontal planes in a uniform height. The fluid is heated from below and is cooled at the top, which results in a convective flow.
- It is considered a 2D flow, since the regular cell-like convection pattern can be captured in a 2D domain.

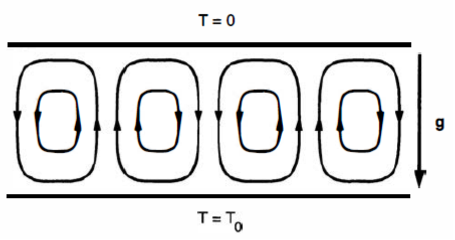

Figure: Heat conduction in an incompressible fluid between horizontal planes

Basic notations

- x and y : the horizontal and vertical directions, respectively
- $y = 0$ at the lower boundary and $y = \pi$ at the upper boundary
- $\mathrm{v} = (v_x, v_y)$: the velocity field
- $T(x, y, t)$: the temperature at the position (x, y) at time t
- The temperature at the lower boundary is $T_0 > 0$, and the temperature at the upper boundary is 0

$$T(y = 0) = T_0 > 0, \quad T(y = \pi) = 0 \tag{1}$$

- q : the heat flux from the convection

$$\mathrm{q} = T\mathrm{v} - \kappa \nabla T, \tag{2}$$

where $\kappa > 0$ is the thermal conductivity

Constitutive equations for the temperature T

(Incompressible flow)

$$\nabla \cdot \mathrm{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{3}$$

(The continuity equation)

$$\frac{\partial T}{\partial t} + \nabla \cdot \mathrm{q} = 0 \tag{4}$$

$\Rightarrow$ The heat equation

$$\frac{\partial T}{\partial t} = -\mathrm{v} \cdot \nabla T + \kappa \nabla T \tag{5}$$

$\Rightarrow$ Define **deviation of temperature** $\theta = T - T^*$ where T^* is the solution of (5). θ satisfies the PDE

$$\frac{\partial \theta}{\partial t} = -\mathrm{v} \cdot \nabla \theta + \frac{T_0}{\pi} v_y + \kappa \nabla \theta \tag{6}$$

and the boundary conditions

$$\theta(y = 0) = 0, \theta(y = \pi) = 0. \tag{7}$$

Constitutive equations for the velocity field v

(Equation of motion)

$$\frac{\partial}{\partial t}(\rho \mathrm{v}) + \nabla \cdot (\rho \mathrm{v}\mathrm{v}) - \rho \mathrm{g} - \nabla \cdot \mathbb{T} = 0 \tag{8}$$

where $\mathbb{T}$ refers to the Cauchy stress tensor
(Incompressible flow) (3)
$\Rightarrow$

$$\frac{\partial}{\partial t}(\rho \mathrm{v}) + \nabla \cdot (\rho \mathrm{v}\mathrm{v}) - \rho \mathrm{g} + \nabla p - \nabla \cdot \mathbb{S} = 0. \tag{9}$$

$\Rightarrow$ Define **stream function** ψ so that $\Delta\psi = \zeta$; the vorticity. ζ satisfies the PDE

$$\frac{\partial \zeta}{\partial t} = -\mathrm{v} \cdot \nabla \zeta + c\frac{\partial \theta}{\partial x} + \nu \Delta \zeta. \tag{10}$$

where c is a thermal expansion coefficient and the boundary conditions for ψ are

$$\psi(y=0) = 0, \quad \psi(y=\pi) = 0 \tag{11}$$

Constitutive equations

For any scalar function $f : (x, y) \mapsto f(x, y)$,

$$\mathrm{v} \cdot \nabla f = v_x f_x + v_y f_y = -\psi_y f_x + \psi_x f_y = \frac{\partial(\psi, f)}{\partial(x, y)}. \tag{12}$$

A system of PDEs for θ and $\Delta\psi$

$$\begin{aligned} \frac{\partial \Delta\psi}{\partial t} &= \nu \Delta^2 \psi + c \frac{\partial \theta}{\partial x} - \frac{\partial(\psi, \Delta\psi)}{\partial(x, y)} \\ \frac{\partial \theta}{\partial t} &= \kappa \Delta \theta + \frac{T_0}{\pi} \frac{\partial \psi}{\partial x} - \frac{\partial(\psi, \theta)}{\partial(x, y)} \end{aligned} \tag{13}$$

Substitute $\zeta = \Delta\psi$ from (10).

Dynamical system

Interpretation of $u(x,t)$ in the sense of dynamical systems

u is a function of t, $u : t \mapsto u(t)$, and $u(x,t)$ is identified with the value of $u(t)$ at x,

$$u : t \mapsto u(t); \quad u(t) : x \mapsto u(t)(x) = u(x,t). \tag{14}$$

- Time is the "primary" variable and space is the "secondary" variable.
- $u(t)$ is a function of x, which belongs to a function space X which is infinite dimensional in general.
- X should meet the requirements specified from the original PDEs and boundary conditions.

Changes from the previous setting

- $\frac{\partial}{\partial t} \to \frac{d}{dt}$
- $\frac{\partial}{\partial x} \to$ operations in the function space X
- the PDE to a dynamical system in X

Absolute climate model

Let

$$u = \begin{pmatrix} \psi \\ \theta \end{pmatrix} : t \mapsto u(t), \quad t \in I \tag{15}$$

where ψ and θ are stream function and the deviation of the temperature, respectively. The original PDE (13) can be rewritten for u as below.

the abstract ODE for u

Considering $u : I \to X$, the **abstract ODE for** u is

$$\frac{d(Du)}{dt} = Au + N(u) \tag{16}$$

where

$$Du = \begin{pmatrix} \Delta\psi \\ \theta \end{pmatrix}, \quad Au = \begin{pmatrix} \nu\Delta^2\psi + c\frac{\partial\theta}{\partial x} \\ \kappa\Delta\theta + \frac{T_0}{\pi}\frac{\partial\psi}{\partial x} \end{pmatrix}, \quad N(u) = \begin{pmatrix} -\frac{\partial(\psi,\Delta\psi)}{\partial(x,y)} \\ -\frac{\partial(\psi,\theta)}{\partial(x,y)} \end{pmatrix}. \tag{17}$$

Dimension reduction

The solutions of the linearized system

$$\frac{d(Du)}{dt} = Au \tag{18}$$

are of the form

$$u = \begin{pmatrix} \psi \\ \theta \end{pmatrix} = \begin{pmatrix} \xi(t)\psi_{a,n} \\ \eta(t)\theta_{a,n} \end{pmatrix}, \tag{19}$$

where

$$\begin{aligned} \psi_{a,n} &: (x,y) \mapsto \psi_{a,n}(x,y) = \sin(ax)\sin(ny), \\ \theta_{a,n} &: (x,y) \mapsto \theta_{a,n}(x,y) = \cos(ax)\sin(ny), \end{aligned} \tag{20}$$

for some $a > 0$ and $n = 1, 2, \cdots$. Then, ξ and η satisfy a system of ODEs

$$\begin{aligned} \dot{\xi} &= -\nu(a^2+n^2)\xi + \frac{ac}{a^2+n^2}\eta, \\ \dot{\eta} &= \frac{aT_0}{\pi}\xi - \kappa(a^2+n^2)\eta. \end{aligned} \tag{21}$$

Dimension reduction

We only consider $n = 1$ which is the most dominant term and look for solution in the subspace spanned by the coordinate vector $u_{a,1}$ keeping a free. The nonlinear component in the system (16) induces

$$N(u_{a,1}) = \begin{pmatrix} \frac{\partial(\psi_{a,1}, \Delta\psi_{a,1})}{\partial(x,y)} \\ \frac{\partial(\psi_{a,1}, \theta_{a,1})}{\partial(x,y)} \end{pmatrix} = \frac{1}{2} a \begin{pmatrix} 0 \\ \sin(2y) \end{pmatrix}, \tag{22}$$

and so, include it as a third coordinate function. An approximate solution of (16) is given as

$$u = \begin{pmatrix} \psi \\ \theta \end{pmatrix} = \begin{pmatrix} \xi(t)\psi_{a,1} \\ \eta(t)\theta_{a,1} \end{pmatrix} - \lambda(t) \begin{pmatrix} 0 \\ sin(2y) \end{pmatrix}. \tag{23}$$

And do this process one more time to get a projection on a three-dimensional state space of ξ, η, and λ.

Dimension reduction

The nonlinear system (16) reduces to a system of ODEs for ξ, η, and λ,

$$\begin{aligned} \dot{\xi} &= -\nu(a^2+1)\xi + \frac{ac}{a^2+1}\eta, \\ \dot{\eta} &= \frac{aT_0}{\pi}\xi - \kappa(a^2+1)\eta - a\xi\lambda, \\ \dot{\xi} &= -4\kappa\lambda + \frac{1}{2}a\xi\eta \end{aligned} \tag{24}$$

the Lorenz equations

By reparametrization of variable, we get the Lorenz equations

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\beta z + xy. \end{aligned} \tag{25}$$

Equilibrium solutions

Lorenz equations can be analyzed by their equilibrium solutions which are either equilibrium points or periodic orbits. The critical points of the linearized model are

- $(0, 0, 0)$ for $\rho > 0$, and additionally,
- $C_{\pm} = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$ for $\rho > 1$.

Illustrate the solution behavior under

- the fixed parameters $\sigma = 10$, and $\beta = \frac{8}{3}$,
- $\rho_H = \frac{\sigma+\beta+3}{\sigma-\beta-1} = 2.4737$ as a critical point for $\rho > 1$ when analyzing $C_{\pm}$
- four distinct initial conditions are chosen arbitrarily to have same distance$(= 0.1)$ from the critical point

In the figures, the points for ICs are marked as *, the origin is marked as *, and for $\rho > 1$, $C_{\pm}$ are marked as *.

Initial conditions * near the origin ($\rho = 0.1, 1$)

- For $\rho = 0.1 < 1$, the origin becomes an attractor and is stable.

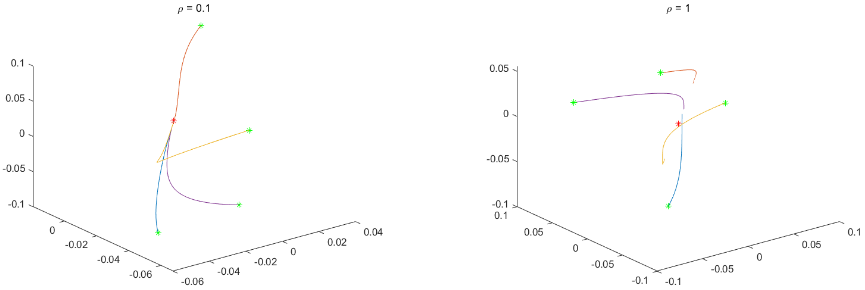

Figure: Solution behavior for initial conditions near the origin

Initial conditions * near the origin ($\rho = 10, 100$)

- When $\rho \geq 1$, the origin is a saddle point.
- For $\rho = 10 > 1$, the solutions converge to $C_{\pm}$. The origin is unstable.
- For $\rho = 100 >> 1$, the solutions are periodic orbits along $C_{\pm}$, and the origin is still unstable.

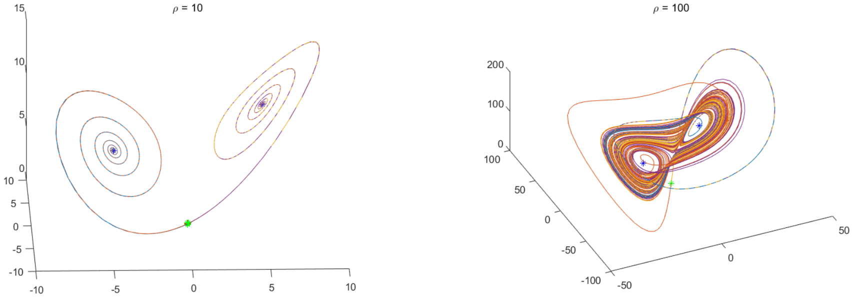

Figure: Solution behavior for initial conditions near the origin

Initial conditions * near C_+ ($\rho = 1.1, 2.3$)

** C_- case is symmetric.*

- For $1 < \rho = 1.1, 2.3 < \rho_H$, C_+ becomes an attractor and is stable.

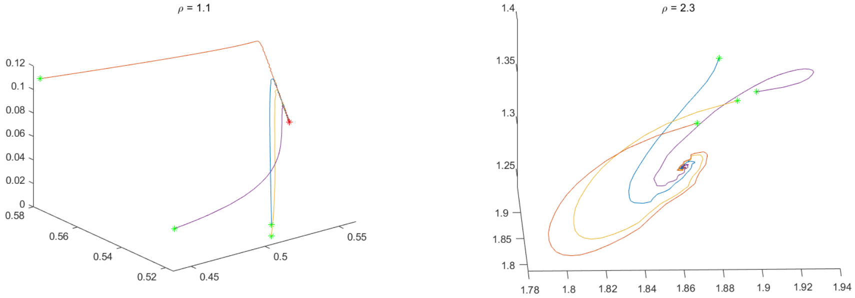

Figure: Solution behavior for initial conditions near C_+

Initial conditions * near C_+ ($\rho = 10, 100$)

** C_- case is symmetric.*

- For $\rho = 10 > \rho_H$, C_+ gets unstable.
- For $\rho = 100 >> 1$, the solutions form periodic orbits along $C_\pm$, and C_+ is still unstable.

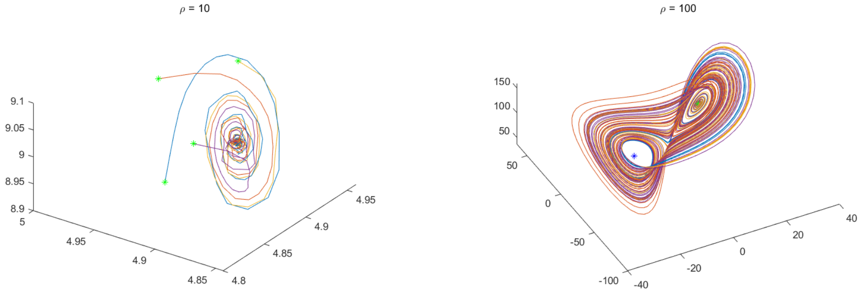

Figure: Solution behavior for initial conditions near C_+

Initial conditions * far from equilibrium points

IC at $(1, 1, 1)$, $\rho = 0.1, 1$.

- For $\rho = 0.1 < 1$, the solution converges to the origin.

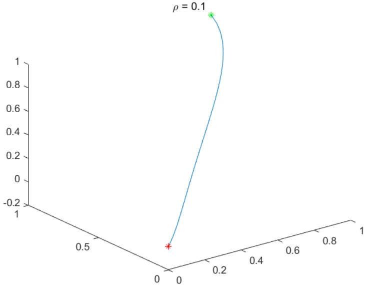

Figure: Solution behavior for initial conditions at $(1, 1, 1)$

Initial conditions * far from equilibrium points

IC at $(1, 1, 1)$, $\rho = 10, 100$.

- For $\rho = 10 > 1$, the solution converges to C_+.
- For $\rho = 100 >> 1$, the solution is a periodic orbit along $C_\pm$ and does not converge.

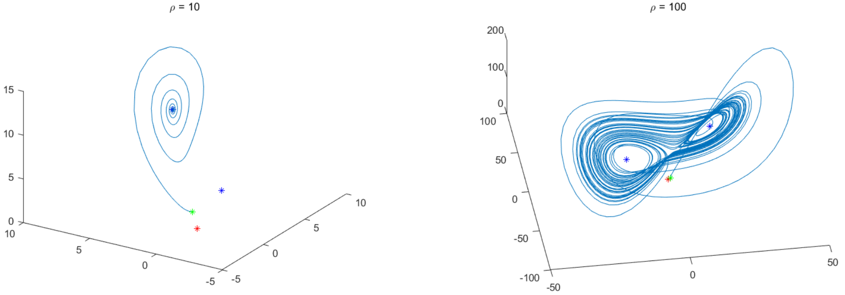

Figure: Solution behavior for initial conditions at $(1, 1, 1)$

Discussion

- In the sense of climate modeling, equilibrium points can be interpreted as a climate state which does not change over multiples of the typical time scale. As for periodic orbits, it can be interpreted as a substantial evidence for time-periodic patterns in the Earth's climate.
- The sensitive dependence to initial condition as well as the choice of parameters manipulate the Earth's atmospheric convection does have the chaotic behavior.
- The current model is a simplified one which is obtained by reducing the dimension of an infinite dimensional function space X truncated by a few dominant terms.
- Instead, we may consider RKHS framework to find an optimal function f which is still in an infinite dimensional space, but easily achievable by a PSD kernel corresponding to the inner product in that space.

References

Lorenz's paper [1], and textbooks [2, 3]

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