Lorenz Equations for Atmospheric Convection Modeling

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Abstract
The Lorenz model is a dynamical system of three first order differential equations. It was designed by Lorenz as a simplified model of atmospheric convection. This project assumes that the Earth's atmosphere is an incompressible fluid situated between two horizontal planes. Using governing equations in 2D hydrodynamics, the steps of Lorenz are followed to derive the Lorenz equations from an abstract climate model. Then, properties of the Lorenz equations are explored and illustrated by individual examples and their interpretations.

Keywords
Lorenz equations — Climate models — Hydrodynamics

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Introduction

Earth climate models consist of various components such as atmosphere, ocean, sea ice, land surface, marine biogeochemistry, ice sheets, and etc. Among climate models, **general circulation models(GCM)**, which characterizes the atmosphere, the land surface, the ocean circulation, and a simplified version of the sea ice, is renowned to describe the climate system the most precisely and complexly.

The hydrodynamic equations are essential to GCMs and they form a system of coupled PDEs. If the system is autonomous, the model can be associated with **a dynamical system in space** X and the variables can be interpreted in a functional sense. General functional spaces satisfying the requirements of the original system of PDEs are infinite dimensional. This makes the analysis of the dynamic system difficult.

Therefore, Lorenz [1] came up with a procedure to reduce the dimensionality of the model. It resulted in a system of three first order differential equations which is called **Lorenz equations** or **Lorenz-63 model**. An analysis of equilibrium solutions of the system can be interpreted in the context of climate science. Numerical experiments with several set of parameters are presented to illustrate the interpretation by examples. [2, 3]

1. Atmospheric Convection Modeling

Lorenz modeled the Earth's atmosphere as an incompressible fluid situated between two horizontal planes in a uniform height as shown in the Figure 1. The fluid is heated from below and is cooled at the top, which results in a convective flow, **Rayleigh-Bernard flow**. The regular cell-like convection pattern can be captured in a two dimensional structure, hence it is enough to assume a two-dimensional flow with one component x which is parallel to the planes and the other component y which is orthogonal to the plane.

1.1 Governing equations

x and y denote the horizontal and vertical directions, respectively, and set $y = 0$ at the lower boundary (at the Earth's surface) and $y = \pi$ at the upper boundary (the tropopause). The velocity field $\mathbf{v} = (v_x, v_y)$ consists of v_x and v_y which are functions of x, y and t.

Let $T(x,y,t)$ be the temperature at the position (x,y) at time t. The temperature at the lower boundary is $T_0 > 0$, and the temperature at the upper boundary is 0,

$$T(y=0) = T_0 > 0, \quad T(y=\pi) = 0. \tag{1}$$

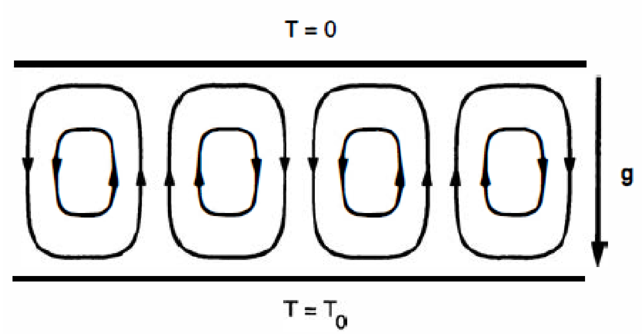

Figure 1. Heat conduction in an incompressible fluid between horizontal planes

Then the heat is transported by convection with the fluid to upward, with the heat flux defined as

$$\mathbf{q} = T\mathbf{v} - \kappa\nabla T, \tag{2}$$

where $\kappa > 0$ is the thermal conductivity.

Here are governing equations in hydrodynamics that formulate the problem. First, the two equations

- (Incompressible flow)

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{3}$$

- (The continuity equation)

$$\frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q} = 0 \tag{4}$$

yield the heat equation

$$\frac{\partial T}{\partial t} = -\mathbf{v} \cdot \nabla T + \kappa\nabla T. \tag{5}$$

When the fluid is at rest and the only heat transfer is due to conduction, this equation (5) has a solution T^* that is merely a linear function of y, given by $T^*(y) = (1 - y/\pi)T_0$. Consider the **deviation of temperature** $\theta = T - T^*$ which satisfies the PDE

$$\frac{\partial \theta}{\partial t} = -\mathbf{v} \cdot \nabla\theta + \frac{T_0}{\pi} v_y + \kappa\nabla\theta \tag{6}$$

and the boundary conditions

$$\theta(y = 0) = 0, \theta(y = \pi) = 0. \tag{7}$$

To obtain the governing equation for the fluid velocity field $\mathbf{v}$, denote the state variables such as the density ρ, the gravity $\mathbf{g}$, and the pressure p which is the negative value of the mean of normal stresses. The governing equation for the fluid velocity field $\mathbf{v}$ can be obtained by

- (Equation of motion)

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) - \rho\mathbf{g} - \nabla \cdot \mathbb{T} = 0 \tag{8}$$

  where $\mathbb{T} = \mathbb{S} - p\mathbb{I}$ refers to the Cauchy stress tensor which is defined by $\mathbb{S} = 2\nu\left(\frac{1}{2}(\nabla\mathbf{v} + \nabla(\mathbf{v})^T) - \frac{1}{3}(\nabla \cdot \mathbf{v})\mathbb{I}\right)$ for the kinematic viscosity ν and the identity tensor $\mathbb{I}$

- (Incompressible flow) (3)

as

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) - \rho\mathbf{g} + \nabla p - \nabla \cdot \mathbb{S} = 0. \tag{9}$$

By assuming the constant density ρ and scaling it to be 1, the fluid velocity is governed by the equation

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \nabla\mathbf{v} - \mathbf{g}' - \nabla p + \nu\Delta\mathbf{v} \tag{10}$$

where $\mathbf{g}'$ refers to the reduced gravity.

A scalar function ψ exists for any divergence-free vector field $\mathbf{v} = (v_x, v_y)$ in two dimensions such that

$$v_x = -\frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \psi}{\partial x} \tag{11}$$

and is called the **stream function**. The laplacian of ψ equals to the vorticity ζ,

Since $v_y = 0$ at the upper and lower boundaries, the stream function should be constant there. Moreover,

$$\psi(y = 0) = 0, \quad \psi(y = \pi) = 0 \tag{12}$$

since the constant but unequal values of ψ at the upper and lower boundaries leads to a nonzero gradient of the stream function and $v_y \neq 0$. Now, the governing equation for the velocity field $\mathbf{v}$ can be written with respect to either of the vorticity ζ or the stream function ψ. Taking the curl of both sides on the equation (10), the vorticity equation is given as

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v} \cdot \nabla\zeta - \nabla \times \mathbf{g}' + \nu\Delta\zeta. \tag{13}$$

The reduced gravity g' is related to the fluid density ρ. By assuming a linear relation between ρ and the temperature T, then the term $\nabla \times \mathbf{g}'$ can be replaced by a constant times $\partial\theta/\partial x$, so the vorticity equation becomes

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v} \cdot \nabla\zeta + c\frac{\partial \theta}{\partial x} + \nu\Delta\zeta. \tag{14}$$

where c is a thermal expansion coefficient.

The nonlinearity in the heat equation (5) and the vorticity equation (14) comes from the terms $\mathbf{v} \cdot \nabla T$ and $\mathbf{v} \cdot \nabla\zeta$, which can be rewritten in terms of the stream function ψ as a Jacobian determinant. For any scalar function $f : (x, y) \mapsto f(x, y)$,

$$\mathbf{v} \cdot \nabla f = v_x f_x + v_y f_y = -\psi_y f_x + \psi_x f_y = \frac{\partial(\psi, f)}{\partial(x, y)}. \tag{15}$$

Finally, a system of PDEs for θ and $\Delta\psi$ is obtained as

$$\begin{aligned}\frac{\partial\Delta\psi}{\partial t} &= \nu\Delta^2\psi + c\frac{\partial\theta}{\partial x} - \frac{\partial(\psi,\Delta\psi)}{\partial(x,y)}\\ \frac{\partial\theta}{\partial t} &= \kappa\Delta\theta + \frac{T_0}{\pi}\frac{\partial\psi}{\partial x} - \frac{\partial(\psi,\theta)}{\partial(x,y)}\end{aligned} \tag{16}$$

by substituting $\zeta = \Delta\psi$.

1.2 Dynamical system

Let u be the vector of all prognostic[1] state variables in a climate model. The state variables vary throughout space and over time, so u is a function of the spatial coordinate x and time t.

In order to view the model as dynamical systems, time and space are considered as "primary" and "secondary" variables, respectively. It can be written that u is a function of t, $u : t \mapsto u(t)$, and $u(x,t)$ is identified with the value of $u(t)$ at x,

$$u : t \mapsto u(t);\;\; u(t) : x \mapsto u(t)(x) = u(x,t). \tag{17}$$

u is defined on the maximal time interval I, and its value $u(t)$ at a point $t \in I$ is a function of x, which belongs to a function space X. X is finite dimensional only if u is independent of x and hence $u(t) \in \mathbb{R}^n$ is a vector. Otherwise, X is an infinite dimensional space of functions.

Now, u is a function which maps a time interval I into X, $u : I \to X$. It reduces partial derivatives with respect to time t to ordinary derivatives with respect to t, and partial derivatives with respect to the spatial variable x to operations in the function space X, and the climate model to a dynamical system in X. X should be defined to meet the requirements specified from the original system of PDEs, boundary conditions, and interface conditions.

An abstract climate model is a dynamical system for the vector u of all the prognostic variables in a function space X,

$$\frac{d(Du)}{dt} = Au + N(u) + F, \tag{18}$$

where D, A, and N represent a simple linear operator, a linear operator with complex structures (may be coupled), and a nonlinear operator, respectively. By shifting the origin, it can be considered that $N(0) = 0$. The term F represents external forces such as turbulent diffusion. F is ignored in this context.

Recall that in the previous section, the system of governing equations (16) can be written in the form (18) without the term F by defining

$$u = \begin{pmatrix}\psi\\ \theta\end{pmatrix} : t \mapsto u(t),\;\; t \in I. \tag{19}$$

The values $u(t)$ are functions of the spatial variables x and y which belongs to a function space X where its elements are sufficiently differentiable and satisfy the boundary conditions (7) and (12). Considering $u : I \to X$, the abstract ODE for u is

$$\frac{d(Du)}{dt} = Au + N(u) \tag{20}$$

where

$$Du = \begin{pmatrix}\Delta\psi\\ \theta\end{pmatrix},\;\; Au = \begin{pmatrix}\nu\Delta^2\psi + c\frac{\partial\theta}{\partial x}\\ \kappa\Delta\theta + \frac{T_0}{\pi}\frac{\partial\psi}{\partial x}\end{pmatrix},\;\; N(u) = \begin{pmatrix}-\frac{\partial(\psi,\Delta\psi)}{\partial(x,y)}\\ -\frac{\partial(\psi,\theta)}{\partial(x,y)}\end{pmatrix}. \tag{21}$$

Current abstract climate model is a dynamical system in a infinite dimensional function space X, which makes it hard to analyze the model. Therefore, it will be further reduced to finite dimensional space by finding some coordinates of $u \in X$ which dominate the dynamics in the later section.

1.3 Dimension reduction

1.3.1 Linearized system

First consider the linearized system

$$\frac{d(Du)}{dt} = Au. \tag{22}$$

The solutions are of the form

$$u = \begin{pmatrix}\psi\\ \theta\end{pmatrix} = \begin{pmatrix}\xi(t)\psi_{a,n}\\ \eta(t)\theta_{a,n}\end{pmatrix}, \tag{23}$$

where

$$\begin{aligned}\psi_{a,n} &: (x,y) \mapsto \psi_{a,n}(x,y) = \sin(ax)\sin(ny),\\ \theta_{a,n} &: (x,y) \mapsto \theta_{a,n}(x,y) = \cos(ax)\sin(ny),\end{aligned}\;t \tag{24}$$

for some $a > 0$ and $n = 1, 2, \cdots$ and $\xi(t)$, and $\eta(t)$ depends on a, and n. By putting the solution in the linearized system (22), it can be shown that ξ and η satisfy a system of ODEs

$$\begin{aligned}\dot{\xi} &= -\nu(a^2+n^2)\xi + \frac{ac}{a^2+n^2}\eta,\\ \dot{\eta} &= \frac{aT_0}{\pi}\xi - \kappa(a^2+n^2)\eta.\end{aligned} \tag{25}$$

This is a planar dynamical system which is stable if and only of both eigenvalues are negative. By checking the determinant, it can be observed that as T_0 increases, the system loses stability. Then, the convective flow pattern emerges at the point $T_0 = T_{a,n} = \frac{\kappa\nu\pi(a^2+n^2)^3}{a^2 c}$. Fixing a, $T_{a,n}$ is the smallest when $n = 1$, so it is reasonable to consider the condition $T_0 > T_{a,1}$.

1.3.2 Nonlinear system

Now, returning to the nonlinear system (20), look for solution in the subspace spanned by the coordinate vector $u_{a,1}$ keeping a free.

$$N(u_{a,1}) = \begin{pmatrix}\frac{\partial(\psi_{a,1},\Delta\psi_{a,1})}{\partial(x,y)}\\ \frac{\partial(\psi_{a,1},\theta_{a,1})}{\partial(x,y)}\end{pmatrix} = \frac{1}{2}a\begin{pmatrix}0\\ \sin(2y)\end{pmatrix} \tag{26}$$

[1] Density, hydrodynamic velocity components, temperature, salinity, and so on.

arises from the nonlinear component in the system. Since neither the Laplace operator Δ nor the partial derivative $\partial/\partial x$ applies to this new component, include it as a third coordinate function and then truncate the expansion. An approximate solution of (20) is given as

$$u = \begin{pmatrix} \psi \\ \theta \end{pmatrix} = \begin{pmatrix} \xi(t)\psi_{a,1} \\ \eta(t)\theta_{a,1} \end{pmatrix} - \lambda(t)\begin{pmatrix} 0 \\ sin(2y) \end{pmatrix}. \tag{27}$$

The essentially new term is

$$\begin{aligned} \frac{\partial(\psi_{a,1}, \sin(2y))}{\partial(x,y)} &= 2a\cos(ax)\sin y\cos(2y) \\ &= a\cos(ax)(\sin(3y) - \sin y) \\ &= a(\cos(ax)\sin(3y) - \theta_{a,1}). \end{aligned} \tag{28}$$

Ignore the term $\cos(ax)\sin(3y)$ which is outside the current coordinate system to get a projection on a three-dimensional state space. By putting the approximate solution (27) to the nonlinear system (20), finally a system of ODEs for ξ, η, and λ,

$$\begin{aligned} \dot{\xi} &= -\nu(a^2+1)\xi + \frac{ac}{a^2+1}\eta, \\ \dot{\eta} &= \frac{aT_0}{\pi}\xi - \kappa(a^2+1)\eta - a\xi\lambda, \\ \dot{\xi} &= -4\kappa\lambda + \frac{1}{2}a\xi\eta \end{aligned} \tag{29}$$

is obtained.

In the following section, it will be checked that the system of three first order differential equations can be written in the form of Lorenz equations by rescaling.

2. Lorenz Equations

2.1 Lorenz equations and atmospheric convection

2.1.1 Description of the equations

A system of three nonlinear autonomous differential equations,

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\beta z + xy \end{aligned} \tag{30}$$

is called the Lorenz equations or the Lorenz-63 model. For a certain parameter set, the solution of the system of equations (30) never repeats its past history exactly and all approximate repetitions have finite duration; that is known as **chaotic dynamics**. Moreover, there are certain structures in the state space, called **strange attractors**. See Figure 2.

2.1.2 Application to atmospheric convection modeling

In the previous section, an abstract climate model without a forcing term (20) yielded a system of ODEs (29). This system can be written of the form of Lorenz equations (30) by

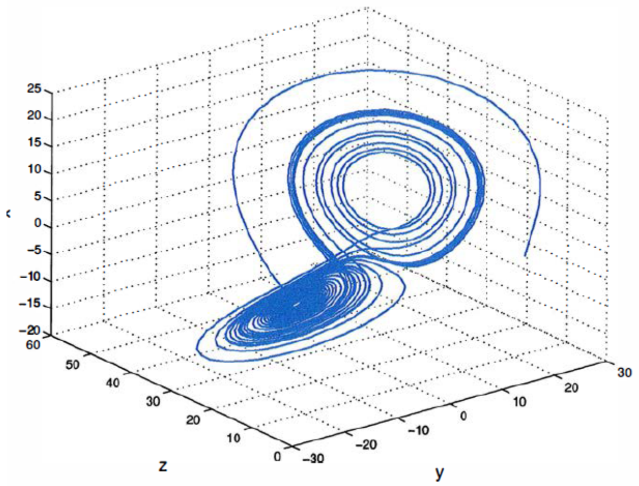

Figure 2. Orbit of the Lorenz equations for $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$

rescaling variables. Rescale $t' = t/\tau$, $\dot{f} = df/dt'$, $\xi = \alpha_1 x$, $\eta = \alpha_2 y$, and $\lambda = \alpha_3 z$, by setting

$$\begin{aligned} \tau &= \frac{1}{\kappa(a^2+1)}, \quad \alpha_1 = \frac{\kappa(a^2+1)\sqrt{2}}{a}, \\ \alpha_2 &= \frac{\kappa\nu(a^2+1)^3\sqrt{2}}{a^2c}, \quad \alpha_3 = \frac{\kappa\nu(a^2+1)^3}{a^2c}. \end{aligned} \tag{31}$$

Then the Lorenz equation (30) is obtained with

$$\begin{aligned} \sigma &= \frac{\nu}{\kappa}, \\ \rho &= \frac{aT_0}{\kappa\pi(a^2+1)}\frac{\alpha_1}{\alpha_2} = \frac{a^2cT_0}{\kappa\nu\pi(a^2+1)^3} = \frac{T_0}{T_{a,1}}, \\ \beta &= \frac{4}{a^2+1}. \end{aligned} \tag{32}$$

In the atmospheric convection modeling, three state variables x, y, and z in the Lorenz equations (30) represent the spatial average of the hydrodynamic velocity, temperature, and temperature gradient, respectively. The dimensionless constants σ, ρ are related to the Prandtl number and Rayleigh number of the fluid, and β is a constant related to the aspect ratio of the domain of interest. All three parameters are positive; σ and β are usually kept fixed with $\sigma > 1+\beta$, and ρ is varied.

2.2 Equilibrium solutions

Dynamical systems can be analyzed by their equilibrium solutions which are either equilibrium points or periodic orbits. In the sense of GCM, equilibrium points can be interpreted as a climate state which does not change over multiples of the typical time scale. Especially, when an equilibrium solution is surrounded by a basin of attraction, either by internal dynamics or by external forcing, it will tend to the equilibrium state at the point. As for periodic orbits, it can be interpreted as a substantial evidence for time-periodic patterns in the Earth's climate.

The equilibrium solutions of the Lorenz equations (30) vary with the range of ρ. If $0<\rho<1$, the Lorenz equations have only one equilibrium solution, the critical point $(0,0,0)$. At this point, the state is with zero-velocity and a linear temperature gradient. In a neighborhood of the origin, the linearized system is

$$\begin{aligned}\dot{x}&=-\sigma x+\sigma y,\\ \dot{y}&=\rho x-y,\\ \dot{z}&=-\beta z.\end{aligned} \tag{33}$$

The z-component is decoupled, and since $\beta>0$, every solution decays to $z=0$ as $t\to\infty$. The x and y components satisfy the system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}-\sigma & \sigma\\ \rho & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=A\begin{pmatrix}x\\ y\end{pmatrix}. \tag{34}$$

The eigenvalues of A are real and negative, and so the origin is the unique positive attractor.

As ρ increases through the value 1, the leading eigenvalue λ_+ increases through the origin and a bifurcation occurs. If $\rho>1$, $\lambda_-<0<\lambda_+$, and hence the origin is a saddle. The origin is no longer stable as $t\to\infty$, and the unstable manifold is spanned by the eigenvector corresponding to the positive eigenvalue λ_+. There are two more equilibrium solutions for $\rho>1$, which are the critical points given as

$$C_\pm=(\pm\sqrt{\beta(\rho-1)},\pm\sqrt{\beta(\rho-1)},\rho-1). \tag{35}$$

Let $\mathbf{x}=(\xi,\ \eta,\ \zeta)^T$. The linearized system in the neighborhood of C_+ is

$$\begin{aligned}\dot{\mathbf{x}}&=\begin{pmatrix}-\sigma & \sigma & 0\\ 1 & -1 & -\sqrt{\beta(\rho-1)}\\ \sqrt{\beta(\rho-1)} & \sqrt{\beta(\rho-1)} & -\beta\end{pmatrix}\mathbf{x}\\ &=A\mathbf{x}.\end{aligned} \tag{36}$$

For $1<\rho<\rho_H=\frac{\sigma+\beta+3}{\sigma-\beta-1}$, the eigenvalues of A are all in the left half of the complex plane, and the linearized system is stable at C_+, but as $\rho>\rho_H$, the real part of the eigenvalues of A changes sign, and hence the linearized system is unstable. Also, a Hopf bifurcation occurs. For $\rho\approx\rho_H$, there are periodic solutions centered roughly at C_+ and they are unstable.

Similar arguments apply to the equilibrium solutions corresponding to C_-. Therefore, the stability is lost near $C_\pm$ as $\rho>\rho_H$, leading to the dynamical system to behave oscillatory.

2.3 Numerical experiments

Following examples are simulated using the ode solver 'ode45' in MATLAB under the fixed parameters $\sigma=10$, and $\beta=\frac{8}{3}$ in the Lorenz equations (30). $\rho_H=\frac{\sigma+\beta+3}{\sigma-\beta-1}=2.4737$ is immediately obtained. Unless specified, four distinct initial conditions are chosen arbitrarily to have same distance($=0.1$) from the critical point. In the figures, the points for ICs are marked as *, the origin is marked as *, and for $\rho>1$, $C_\pm$ are marked as *.

Initial conditions near the origin $\rho=0.1,1,10,100$ are tested in Figure 3. For $\rho=0.1<1$, the origin becomes an attractor and is stable. When $\rho\geq1$, the origin is no more an attractor, and is a saddle point. For $\rho=10>1$ but a relatively small value, the solutions starting from points near the origin converge to $C_\pm$. Small perturbation on the initial conditions leads to convergence to different points, and so the origin is unstable. For $\rho=100>>1$, the solutions starting from points near the origin are periodic orbits along $C_\pm$, and the origin is still unstable.

Initial conditions near C_+ Since $\rho>1$ is relevant to $C_\pm$ and $\rho_H=2.4737$ is critical, $\rho=1.1,2.3,10,100$ are tested in Figure 4. For $1<\rho=1.1,2.3<\rho_H$, C_+ becomes an attractor and is stable. For $\rho=10>\rho_H$, C_+ gets unstable. But, for $\rho=100>>1$, the solutions form periodic orbits along $C_\pm$, and C_+ is still unstable. *Since the system is symmetric with respect to the reflection along the z-axis, initial conditions near C_- may have similar behavior.*

Initial conditions far from equilibrium points Given an initial condition at $(1,1,1)$, $\rho=0.1,1,10,100$ are tested in Figure 5. For $\rho=0.1<1$, the solution converges to the origin. For $\rho=10>1$, the solution converges to C_+. For $\rho=100>>1$, the solution is a periodic orbit along $C_\pm$ and does not converge.

3. Discussion

From the simulation, it turned out that the solution of the Lorenz equations is sensitive to initial condition as well as the choice of parameters. This implies the Earth's atmospheric convection does have the chaotic behavior.

In fact, the current model is a simplified one which is obtained by reducing the searching domain from an infinite dimensional function space X to a finite dimensional space which is spanned by a few dominant terms. Instead, we may consider a PSD kernel which can be cheaply computable, and corresponds to the inner product of the infinitely many basis terms. Then, by properly choosing an objective function which accounts for the required conditions of the PDE solutions, we may have a different approach to the problem as solving an optimization problem on a function space generated by the kernel.

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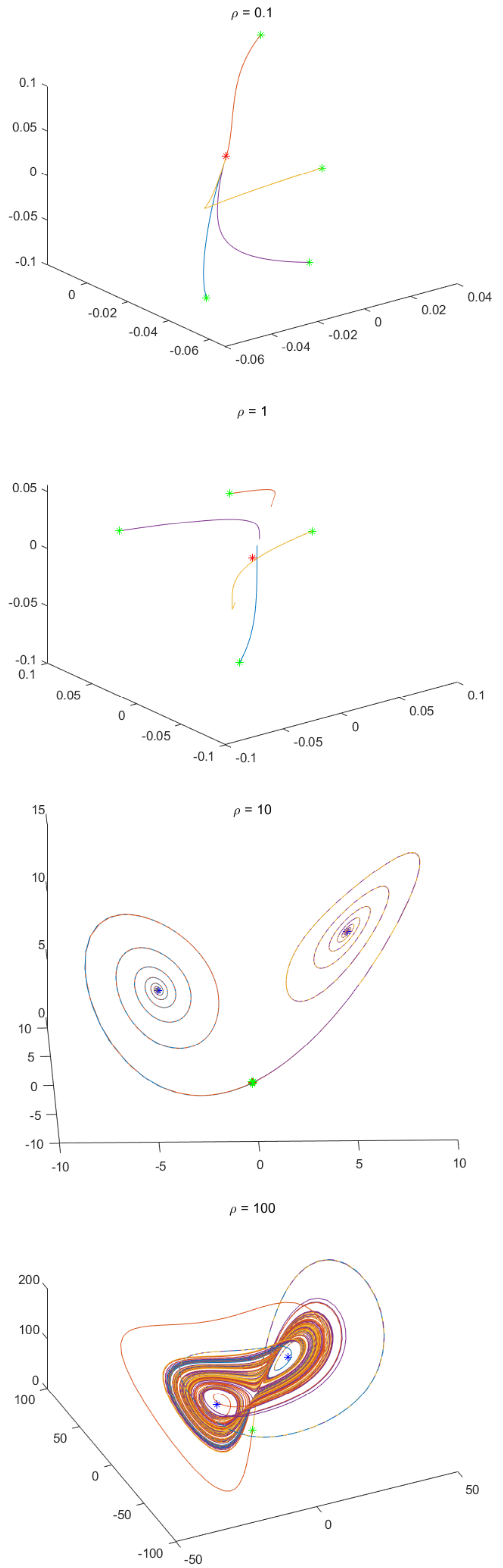

Figure 3. Solution behavior for initial conditions near the origin

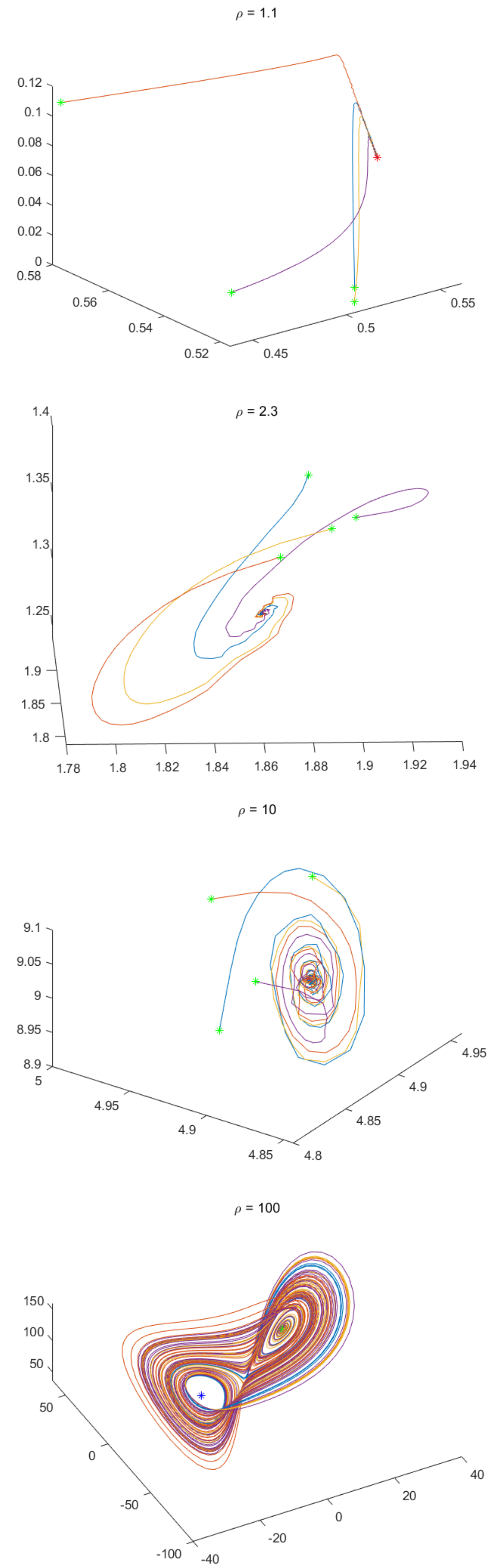

Figure 4. Solution behavior for initial conditions near C_+

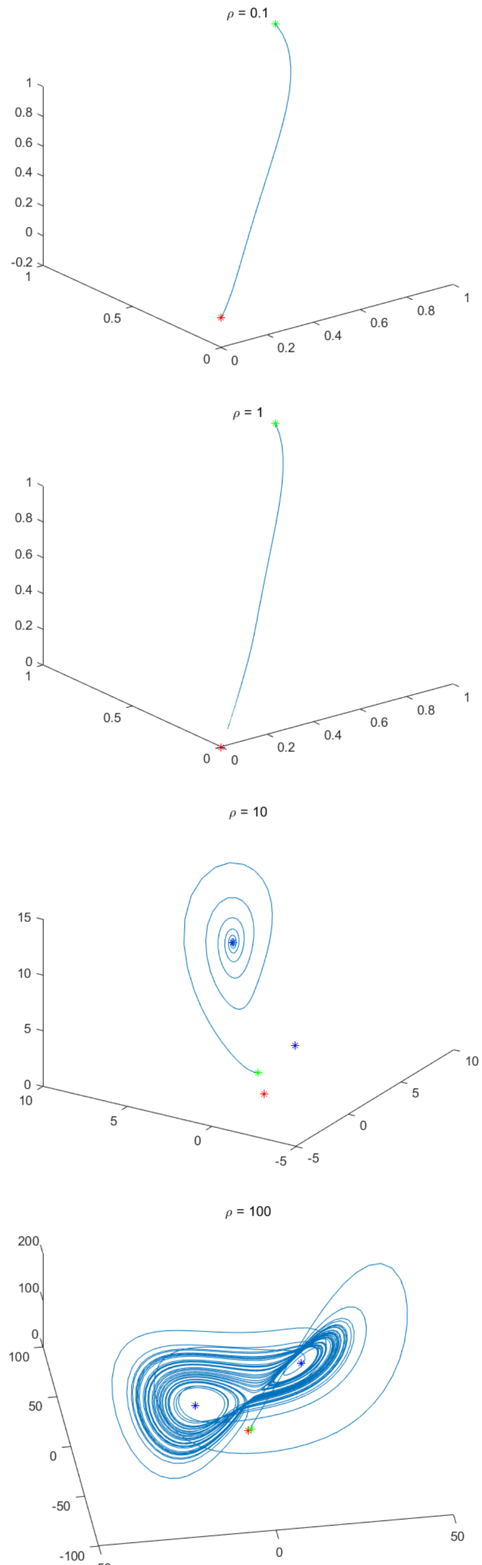

Figure 5. Solution behavior for initial conditions at $(1,1,1)$